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Irreducible embeddings and polynomial tensors

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Abstract. The problem of finding irreducible tensors which are polynomials in the components of an irreducible tensor (n) of a compact semisimple group H is simplified if the representation (n) is contained irreducibly in a representation (m) of a larger group G of which H is a subgroup. A number of examples are given for which the problem is completely solved by generating function methods. We discuss how one can use the computer to tackle the more difficult cases; the problem of the reduction of the SO(8) enveloping algebra is given as an example.

1. Introduction

A problem which arises in many contexts in mathematical physics is that of determining a basis for all irreducible tensors whose components are polynomials in the components of a given tensor Γ which transforms by an irreducible representation (n) of a compact semisimple Lie group H. In other words, the problem consists of decomposing the symmetric tensor product of p *identical* copies $(p=1,\infty)$ of Γ into a direct sum of irreducible tensors of H; this is known as the calculation of the symmetric plethysm and these tensors are referred to as polynomial tensors based on Γ.

In nuclear physics, polynomial tensors have been used as states to describe quadrupole and octupole nuclear vibrations (Chacon *et al* 1976, Gaskell *et al* 1978, Rohozinski 1978, Vanden Berghe and De Meyer 1979). The problem of constructing states for even higher modes has also been considered (Rohozinski and Greiner 1980, Bystricky *et al* 1982). The knowledge of a basis for polynomial tensors has proven useful in the studies of bifurcations in geophysics and elasticity theory (Sattinger 1978, Chossat 1979). Since there exists a one-to-one correspondence between a basis for irreducible tensor operators in the enveloping algebra of a Lie group H and a basis for polynomial tensors based on a tensor Γ that transforms by the adjoint representation of H, polynomial tensors constitute an efficient way of tackling the problem of the reduction (decomposition) of enveloping algebras. The enveloping algebra has proven to be a useful mathematical concept in theoretical physics; its structure has been the object of many investigations. Recently, in the context of the interacting boson model in nuclear physics, a systematic study of a basis for symmetry-conserving higher-order interaction terms was made exploiting generating function (GF) techniques and the concept of enveloping algebra and degenerate enveloping algebra (Van der Jeugt and De Meyer 1987). Earlier, the introduction of such higher-order terms in the Hamiltonian was reported (Vanden Berghe *et al* 1985) to give rise to a much better approximation of the energy spectrum. A complete set of SU(3) tensor operators was recently

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defined in the enveloping algebra of SO(8) (Biedenharn and Flath 1984, Le Blanc and Rowe 1987); this is of particular importance for the computation of Wigner and Racah coefficients in the studies of nuclear collective motion.

The solution to the problem of finding a basis for polynomial tensors based on a tensor Γ is most conveniently effected, and presented, by finding the GF $F(U; N) \equiv F(U; N_1, \ldots, N_l)$ which is a fraction or sum of fractions, whose denominator factors have the form $(1-X)$, the X and the numerator terms being algebraic monomials in the variables U, N_1, $N_2, \ldots, N_l$, where l is the rank of the group H (the dimension of its weight space). Consider the power expansion of F:

$$F(U; N_1, \ldots, N_l) = \sum_u U^u \sum_n c_{un} N^n \qquad N^n \equiv \prod_{i=1}^{l} N_i^{n_i}. \tag{1.1}$$

Then U carries the degree u and N_i the representation label n_i as exponents; the presence of a term $U^u\, c_{un} N^n$ in (1.1) informs us that the number of irreducible polynomial tensors that transform by the irreducible representation $(n) \equiv (n_1, n_2, \ldots, n_l)$, of degree u in Γ, is c_{un}. Throughout this paper, all representation labels are Cartan, or Dynkin, labels and the order of labels in the l-plet (n) follows the numbering of the simple roots given in McKay and Patera (1981). The GF $F(U; N)$ suggests an integrity basis, a finite number of 'elementary' polynomial tensors, in terms of which all tensors described by the GF can be expressed as stretched products (representation labels additive), with certain products being forbidden.

An algorithm for evaluating $F(U; N)$ starting with the corresponding GF for weights has been proposed (Gaskell *et al* 1978). This method is extremely tedious to implement for a group of high rank or for a tensor Γ of high dimension. Another approach which has been used to solve this problem is based on a well known relation (Weyl 1946) between irreducible representations of the symmetric group and those of the group $\mathrm{U}(d)$ of unitary $d \times d$ matrices. It can be shown (Patera and Sharp 1980) that, given a tensor Γ of dimension d that transforms by an irreducible representation (n), the problem of finding the symmetric plethysm is equivalent to that of finding the branching rules of $\mathrm{SU}(d) \supset \mathrm{H}$ restricted to the symmetric representations of $\mathrm{SU}(d)$ where the representation $(1, 0, \ldots, 0)$ of $\mathrm{SU}(d)$ contains (n) once. The task of finding the GF for these branching rules can be quite difficult; it is important to take advantage of any simplification which presents itself.

Such a simplification occurs when H is not maximal in $\mathrm{SU}(d)$ but occurs in a chain $\mathrm{H} \subset \mathrm{G} \subset \mathrm{SU}(d)$ and when the representation (n) is embedded irreducibly in a representation (m) of G such that $(n) \subset (m) \subset (1, 0, 0, \ldots 0)$. The problem 'factors' into two simpler problems:

(*a*) finding the GF $J(U; M) \equiv J(U; M_1, \ldots M_{l^*})$ for G-polynomial tensors based on a tensor that transforms by (m); l^* is the rank of G;

(*b*) finding the GF $K(M; N) = K(M_1, \ldots M_{l^*}; N_1, \ldots, N_l)$ for branching rules from G to H where l is the rank of H; the complete GF K is not always needed; M_i which do not appear in $J(U, M)$ can be set equal to zero. To find the desired GF $F(U; N)$ it is necessary to combine $K(M; N)$ with $J(U; M)$ (or 'substitute' K into J):

$$F(U; N) = J(U; M) K(M^{-1}; N)|_{M^0} \tag{1.2}$$

where $|_{M^0}$ is an instruction to retain only the terms of degree zero in the variables M_i. The number of denominator terms of the GF $F(U; N)$ resulting from (1.2) is equal to the number for $J(U; M)$ plus the number for $K(M; N)$ less the number of M_i to be eliminated by $|_{M^0}$.

Dynkin (1957) has given a complete list of irreducible embeddings of a subgroup representation into a group representation. Our purpose in this paper is to illustrate how one can exploit Dynkin's work to calculate GF for polynomial tensors or at least get part of their integrity basis when the basis is large. In § 3 we treat all the 'tractable' cases in Dynkin's list; roughly, 'tractable' means that terms in the GF have no more than twelve denominator factors.

The case in which a group representation reduces under a subgroup to an irreducible representation and a scalar can also be used to construct GF for polynomial tensors. In that case (1.2) is replaced by

$$F(U, N) = (1 - U)J(U; M)K(M^{-1}; N)|_{M^0} \tag{1.3}$$

a few examples of which are treated in § 4. The embeddings were found in the branching rule tables of McKay and Patera (1981). The stratagem described above works equally when the group H is a subjoined group of G, rather than a subgroup. An example is given in § 5.

The GF $J(U; M)$ and $K(M; N)$ are usually determined by first finding the corresponding integrity basis and syzygies; this is done by proceeding systematically through the irreducible representations of the parent group, starting with representations for which only one representation label is non-zero and systematically increasing the number of non-zero labels. For every representation of the parent group one considers, one has to take all possible products of powers of the elementary multiplets (taking into account the syzygies) and check whether it gives the correct branching by making a dimension and second-order index check. In cases where the integrity basis is large, this procedure can be very tedious. A computer program MULTI has been written which does all this work, leaving the user with the task of guessing the elementary multiplets and their syzygies. Knowledge of the branching rules reduces the guesswork considerably. For low-dimensional representations of the parent group, these can be obtained from tables like those of McKay and Patera; if one needs to consider high-dimensional representations, in many cases these can be obtained by Young diagram techniques and the whole process can be computerised. In order to illustrate how one can use the computer to tackle cases where the integrity basis is large, we consider in § 6 the problem of the reduction of the SO(8) enveloping algebra. We now give rules for determining the maximum number of denominator factors in any given term of a GF; the knowledge of such a number provides important clues for the construction of GF and is a measure of how difficult it will be to determine.

2. Determination of the number of denominator factors

The number of denominator factors in a typical term of the GF for polynomial tensors based on a tensor Γ of a group H is $f_p = d - i$, where d is the dimension of Γ and i is the number of internal labels required by the representations of H described by the GF. For i we have (Seligman and Sharp 1983)

$$i = \tfrac{1}{2}(r_h - l_h - r_j + l_j) \tag{2.1}$$

where r_h and l_h are the order (number of generators of its algebra) and rank of H, and r_j and l_j are the order and rank of the subgroup J whose Dynkin diagram is that corresponding to the zero labels of the possibly degenerate representations appearing in the GF. Let us illustrate this point with a few examples. The GF for SU(n) polynomial tensors based on the tensor $(2, 0, \ldots, 0)$ is

$$F(U; N_1, N_2, \ldots, N_{n-1}) = [(1-UN_1^2)(1-U^2N_2^2)(1-U^3N_3^2)\ldots \times(1-U^{n-1}N_{n-1}^2)(1-U^n)]^{-1}. \tag{2.2}$$

This is an example of a non-degenerate case since all $(n-1)$ SU(n) representation labels (the exponents of the N_i) appear in the GF and therefore $r_{\rm j}=l_{\rm j}=0$; $d=(n+1)n/2$, $r_{\rm h}=n^2-1$ and $l_{\rm h}=n-1$ so $d-i=n$ which agrees with (2.2). As a second example we consider the GF for SU$(2n)$ polynomial tensors based on a tensor $(0, 1, 0, \ldots, 0)$:

$$F(U; N_2, N_4, \ldots, N_{2n-2}) = [(1-UN_2)(1-U^2N_4)\ldots(1-U^{n-1}N_{2n-2})(1-U^n)]^{-1}. \tag{2.3}$$

This is a degenerate case since only tensors of the type $(0, n_2, 0, n_4, 0, \ldots, n_{2n-2}, 0)$ appear in (2.3). The group J whose Dynkin diagram correspond to the zero labels is simply SU(2) × SU(2) × ... × SU(2) n times; therefore $r_{\rm j}=3n$ and $l_{\rm j}=n$. The dimension of the tensor $(0, 1, 0, \ldots, 0)$ is $d=n(2n-1)$, $r_{\rm h}=4n^2-1$ and $l_{\rm h}=2n-1$. The number of denominator factors is $f_p=d-i=n$ which agrees with (2.3).

Let us now consider the case of GF for branching rules. The number of denominator factors in a GF for branching rules from a group G to a subgroup H is $f_{\rm b}=c_{\rm g}+i_{\rm g}-i_{\rm h}$; here $c_{\rm g}$ is the number of non-vanishing G-representation labels, $i_{\rm g}$ and $i_{\rm h}$ are the required internal labels of the most general representations of G and H appearing in the GF; $i_{\rm g}$ and $i_{\rm h}$ are calculated using (2.1). As a first example let us consider the GF for branching rules (Sharp 1970) of Sp(6) ⊃ Sp(4) × SU(2):

$$K(M_1, M_2, M_3; N_1, N_2, N_3) = [(1-M_1N_1)(1-M_1N_3)(1-M_2N_2)(1-M_2)(1-M_3N_1)(1-M_3N_2N_3)]^{-1} \times[(1-M_2N_1N_3)^{-1}+M_1M_3N_2(1-M_1M_3N_2)^{-1}] \tag{2.4}$$

where M_1, M_2 and M_3 carry the Sp(6) labels as exponents, N_1 and N_2 those of Sp(4) and N_3 carries the SU(2) label (the dimension of an SU(2) representation j is $j+1$). For Sp(6) $r_{\rm g}=21$, $c_{\rm g}=3$ and since all representation labels of Sp(6) appear in (2.4) we have $r_{\rm j}=0$ and $l_{\rm j}=0$ so $i_{\rm g}=9$. For Sp(4) × SU(2), $r_{\rm h}=13$, $l_{\rm h}=3$ and $r_{\rm j}=l_{\rm j}=0$ so $i_{\rm h}=5$; therefore $f_{\rm b}=7$, which agrees with (2.4). As a second example we consider the branching rules of SU(36) ⊃ SU(9) restricted to the symmetric representations of SU(36); the GF is

$$K(M_1; N_2, N_4, N_6, N_8) = [(1-M_1N_2)(1-M_1^2N_4)(1-M_1^3N_6)(1-M_1^4N_8)]^{-1} \tag{2.5}$$

where M_1 carries the SU(36) label and the N_i those of SU(9); (2.5) is an example of a degenerate case since only the SU(36) representations of the type $(m_1, 0, \ldots, 0)$ and SU(9) representations of the type $(0, n_2, 0, n_4, 0, n_6, 0, n_8)$ appear in (2.5). For SU(36) $c_{\rm g}=1$, $r_{\rm g}=1295$, $l_{\rm g}=35$, $r_{\rm j}=1224$ and $l_{\rm j}=34$ (here J = SU(35)) so that $i_{\rm g}=35$. For SU(9), $r_{\rm h}=80$, $l_{\rm h}=8$, $r_{\rm j}=12$ and $l_{\rm j}=4$ (since J = SU(2) × SU(2) × SU(2) × SU(2)) so $i_{\rm h}=32$; the number of denominator factors should therefore be $f_{\rm b}=4$, which agrees with (2.5). The GF (2.5) is a particular case (M_1 replaced by U) of the following GF for SU$(2n+1)$ polynomial tensors based on a tensor $(0, 1, 0, \ldots, 0)$:

$$F(U; N_2, N_4, \ldots, N_{2n}) = [(1-UN_2)(1-U^2N_4)\ldots(1-U^nN_{2n})]^{-1} \tag{2.6}$$

where U carries the degree and the N_i the SU$(2n+1)$ representation labels. We now proceed through Dynkin's list (Dynkin 1957).

3. Polynomial tensors based on a tensor

3.1. Sp(2n) polynomial tensors based on a tensor (k, 0, . . . , 0)

The dimension of the tensor $\Gamma \equiv (k, 0, \ldots, 0)$ is $d = \binom{2n-1+k}{k}$; the GF for a polynomial tensors based on Γ is equal to the GF for branching rules of Sp$(2n) \subset$ SU(d) restricted to the symmetric representations of SU(d). The task of constructing such a GF may be reduced by exploiting the following irreducible embedding:

$$\begin{aligned} &C_n \subset A_{2n-1} \qquad n \geqslant 2, k \geqslant 2 \\ &(k, 0, \ldots, 0) \subset (k, 0, \ldots, 0). \end{aligned} \tag{3.1}$$

The problem factors into two simpler problems. First we construct a GF $J(U; M)$ for SU$(2n)$ polynomial tensors based on a tensor $(k, 0, \ldots, 0)$ and then construct a GF $K(M; N)$ for branching rules from SU$(2n)$ to Sp$(2n)$; we obtain the final answer by combining both results using (1.2).

The case $k=2$ is of particular interest since the GF for polynomial tensors based on a tensor $(2, 0, \ldots, 0)$ describes the reduction of the enveloping algebra of Sp$(2n)$. The GF $J(U; M)$ is given in (2.2); the maximum number of denominator factors (see § 2) in any term of $K(M; N)$ is $f_b = n^2+n-1$ ($c_g = 2n-1$, $i_g = n(2n-1)$ and $i_h = n^2$). For $n=2$

$$J(U; M_1, M_2, M_3) = [(1-U^4)(1-UM_1^2)(1-U^2M_2^2)(1-U^3M_3^2)]^{-1} \tag{3.2}$$

and the GF for branching rules from SU(4) to Sp(4) is (Patera and Sharp 1980)

$$\begin{aligned} &K(M_1, M_2, M_3; N_1, N_2) \\ &\quad = [(1-M_1N_1)(1-M_2N_2)(1-M_2)(1-M_3N_1)(1-M_1M_3N_2)]^{-1} \end{aligned} \tag{3.3}$$

where the M_i and N_i carry the SU(4) and Sp(4) representation labels respectively; notice that $f_b = 5$, which agrees with (3.3). The GF $F(U; N_1, N_2)$ for Sp(4) polynomial tensors based on a tensor (2, 0) is then obtained by 'substituting' (3.3) into (3.2) using the method proposed in (1.2). The GF (3.2) informs us that only SU(4) representations with even labels need to be considered; keeping only terms which are even powers in the M_i in (3.3) and denoting the resulting GF by $KE(M; N)$, the operation (1.2) is easily effected by making the following substitutions in $KE(M; N)$: $M_1^2 \to U$, $M_2^2 \to U^2$ and $M_3^2 \to U^3$. We then get (Couture and Sharp 1980)

$$\begin{aligned} F(U; N_1, N_2) = [&(1-U^2)(1-U^4)(1-UN_1^2)(1-U^2N_2^2) \\ &\times (1-U^3N_1^2)(1-U^2N_2)]^{-1}(1+U^4N_1^2N_2). \end{aligned} \tag{3.4}$$

The GF (3.4) also describes the reduction of the enveloping algebra of SO(5) since it is isomorphic to that of Sp(4). The GF describing the decomposition of the Sp(6) enveloping algebra (Couture and Sharp 1980) corresponds to the $n=3$ case; it was also obtained by using the irreducible embedding given in (3.1).

For $k=3$, the task is more difficult; the maximum number of denominator factors for a GF for polynomial tensors based on a tensor $(3, 0, \ldots, 0)$ is $f_p = \frac{2}{3}n(n+1)(2n+1) - n^2$. For $n=2$, $f_p = 16$.

3.2. SO(2n+1) polynomial tensors based on a tensor $(\underbrace{0, 0, \ldots,}_{k} 1, 0, \ldots, 0)$

The dimension of the tensor $\Gamma \equiv (\underbrace{0, 0, \ldots,}_{k} 1, 0, \ldots, 0)$ is $d = \binom{2n+1}{k}$; the task of constructing a GF for such SO$(2n+1)$ polynomial tensors may be simplified if one considers the following irreducible embedding:

$$\begin{aligned} &B_n \subset A_{2n} \qquad n > k \geqslant 2 \\ &(\underbrace{0, 0, \ldots,}_{k} 1, 0, \ldots 0) \subset (\underbrace{0, 0, \ldots,}_{k} 1, 0, \ldots, 0). \end{aligned} \tag{3.5}$$

The problem factors into two simpler ones. Firstly, construct a GF $J(U; M)$ for SU$(2n+1)$ tensors based on a tensor $(\underbrace{0, 0, \ldots,}_{k} 1, 0, \ldots, 0)$; secondly, construct the GF $K(M; N)$ for the branching rules from SU$(2n+1)$ to SO$(2n+1)$. The reduction of the enveloping algebra of SO$(2n+1)$ is described by the $k=2$ case. $J(U, M)$ is given in (2.6); the maximum number of denominator factors in any term of $K(M; N)$ is $f_b = n^2+n$ $(c_g = n,\ i_g = 2n^2,\ i_h = n^2)$. The $n=3$ case corresponds to the GF describing the decomposition of the SO(7) enveloping algebra and has been obtained (Couture and Sharp 1980) using the embedding (3.5). For $n=4$, $f_b = 20$ so the task would be quite difficult.

3.3. SO(2n+1) polynomial tensors based on a tensor (0, 0, ..., 0, 2)

The dimension of the tensor $\Gamma \equiv (0, 0, \ldots, 0, 2)$ is $d = \binom{2n+1}{n}$; the irreducible embedding is

$$\begin{aligned} &B_n \subset A_{2n} \qquad n \geqslant 2 \\ &(0, 0, \ldots, 2) \subset (\underbrace{0, 0, \ldots,}_{n} 1, 0, \ldots, 0). \end{aligned} \tag{3.6}$$

In order to constructs a GF $F(U; N)$ for SO$(2n+1)$ polynomial tensors based on Γ one first constructs a GF $J(U; M)$ for SU$(2n+1)$ polynomial tensors based on the tensor $(\underbrace{0, 0, \ldots,}_{n} 1, 0, \ldots, 0, 0)$; one then constructs the GF $K(M; N)$ for branching rules of SU$(2n+1)$ to SO$(2n+1)$. The maximum number of denominator factors in any term of $F(U; N)$ is $d - n^2$. The case $n = 2$ is interesting since it constitutes another way of calculating (3.4); for $n=2$, $d=10$. The GF $J(U; M)$ for SU(5) polynomial tensors based on a tensor (0, 1, 0, 0) is given by (2.7):

$$J(U; M_2, M_4) = [(1 - UM_2)(1 - U^2M_4)]^{-1}.$$

The GF for branching rules from SU(5) to SO(5) restricted to SU(5) representations, for which only the second and fourth labels are non-zero, is (Patera and Sharp 1980)

$$\begin{aligned} K(M_2, M_4; N_1, N_2) = {} & [(1 - M_2^2)(1 - M_4^2)(1 - M_2N_2^2)(1 - M_4N_1)(1 - M_2M_4N_2^2) \\ & \times (1 - M_2^2N_1^2)]^{-1}[(1 + M_2^2M_4N_1N_2^2)]. \end{aligned}$$

The GF for SO(5) polynomial tensors based on a tensor (0, 2) is obtained through the prescription (1.2), which is easily performed by making the following substitutions in $K(M; N)$: $M_2 \to U$, $M_4 \to U^2$. The GF obtained is equal to the one given in (3.4) if we make the following substitutions: $N_1 \longleftrightarrow N_2$. The task of constructing such GF soon becomes very difficult since, for $n=3$, $d - n^2 = 26$.

3.4. *SO(2n) polynomial tensors based on a tensor* $(0, 0, \underbrace{\ldots}_{k}, 1, 0, \ldots, 0)$

The dimension of the tensor $(0, 0, \underbrace{\ldots}_{k}, 1, 0, \ldots, 0)$ is $d = \binom{2n}{k}$ and the irreducible embedding to be used is

$$D_n \subset A_{2n-1} \qquad n \geq 4, n-1 > k \geq 2$$

$$(0, 0, \underbrace{\ldots}_{k}, 1, 0, \ldots, 0) \subset (0, 0, \underbrace{\ldots}_{k}, 1, 0, \ldots, 0).$$

$J(U; M)$ is the GF for branching rules of $SU(d)$ to $SU(2n)$ restricted to the symmetric representations of $SU(d)$; $K(M; N)$ is the GF for branching rules from $SU(2n)$ to $SO(2n)$ where some of the $SU(2n)$ representation labels may be zero (degenerate case). The case $k=2$ describes the reduction of the $SO(2n)$ enveloping algebra; the $SO(8)$ enveloping algebra is discussed in § 6.

3.5. *SO(2n) polynomial tensors based on a tensor (0, 0, ..., 1, 1)*

The dimension of the tensor $(0, 0, \ldots, 1, 1)$ is $d = \binom{2n}{n-1}$ and the irreducible embedding of interest is

$$D_n \subset A_{2n-1} \qquad n \geq 3$$

$$(0, 0, \ldots, 1, 1) \subset (0, 0, \underbrace{\ldots}_{n-1}, 1, 0, \ldots, 0) \qquad n > 3$$

and

$$(1, 0, 1) \subset (0, 1, 0, 0, 0) \qquad n = 3.$$

Assuming that all representation labels appear in the GF $J(U; M)$, then the maximum number of denominator factors in the GF for branching rules from $SU(2n)$ to $SO(2n)$ is $f_b = n^2 + 2n - 1$ ($c_g = 2n-1$, $i_g = n(2n-1)$, $i_h = n(n-1)$). The solution to the $n=3$ case gives the reduction of the $SU(4)$ enveloping algebra; in this case $J(U; M)$ is given by (2.3). Unfortunately, if one attempts the reduction of any other $SU(n)$ enveloping algebra ($SU(2)$ and $SU(3)$ have already been done), Dynkin's list informs us that there is no irreducible embedding that could be used to simplify the task.

3.6. *SO(2n+1) polynomial tensors based on a tensor (0, 0, ..., k)*

The irreducible embedding of interest is

$$B_n \subset D_{n+1} \qquad k \geq 1, n \geq 3$$

$$(0, 0, \ldots, k) \subset (0, \ldots, k, 0).$$

The GF $K(M; N)$ giving the branching rules of $SO(2n+2)$ to $SO(2n+1)$ is easily constructed since integrity bases for all n are known (Sharp 1970). What remain to be calculated are the GF $J(U; M)$ for $SO(2n+2)$ polynomial tensors based on a tensor $(0, 0, \ldots, k, 0)$. Let us first consider the case $n=3$, $k=1$. Since the dimension of the $SO(8)$ tensor $(0, 0, 1, 0)$ is equal to 8, the GF $J(U; M)$ is equal to the GF for the branching rules of $SU(8)$ to $SO(8)$ restricted to the symmetric representations of $SU(8)$; it is easily shown that

$$J(U; M) = [(1-U^2)(1-UM_3)]^{-1} \tag{3.7}$$

where U carries the SU(8) label and M_3 the third representation label of SO(8). The GF $K(M; N)$ for branching rules from SO(8) to SO(7) restricted to SO(8) representations, for which only the third label is non-zero, is

$$K(M_3; N_3) = (1 - M_3N_3)^{-1} \tag{3.8}$$

where N_3 carries the third representation label of SO(7). Following the prescription (1.2), the GF for SO(7) polynomial tensors based on a (0, 0, 1) tensor is

$$F(U; N_3) = [(1 - U^2)(1 - UN_3)]^{-1}. \tag{3.9}$$

Note that $f_p = 2$, which agrees with (3.9).

For the case $n = 4$, $k = 1$ the GF for SO(10) polynomial tensors based on a tensor (0, 0, 0, 1, 0) is

$$J(U; M_1, M_4) = [(1 - U^2M_1)(1 - UM_4)]^{-1} \tag{3.10}$$

where M_1 and M_4 carry the first and fourth SO(10) representation labels as exponents. The appropriate GF for branching rules from SO(10) to SO(9) is

$$K(M_1, M_4; N_1, N_4) = [(1 - M_1N_1)(1 - M_1)(1 - M_4N_4)]^{-1}. \tag{3.11}$$

'Substituting' (3.11) into (3.10) through the prescription (1.2), we get the GF for SO(9) polynomial tensors based on a tensor (0, 0, 0, 1)

$$F(U; N_1, N_5) = [(1 - U^2)(1 - U^2N_1)(1 - UN_4)]^{-1}. \tag{3.12}$$

$f_p = 3$ ($d = 16$, $r_h = 36$, $l_h = 4$, $r_j = 8$, $l_j = 2$), which agrees with (3.12).

Finally we consider the case $n = 5$, $k = 1$. The GF for SO(12) polynomial tensors based on a tensor (0, 0, 0, 0, 1, 0) is

$$J(U; M_2, M_4, M_5) = [(1 - U^4)(1 - U^2M_2)(1 - U^4M_4)(1 - UM_5)(1 - U^3M_5)]^{-1}$$

where U carries the degree of the tensors and the M_i the second, fourth and sixth SO(12) labels. The GF for branching rules from SO(12) to SO(11) is

$$K(M_2, M_4, M_5; N_1, N_2, N_3, N_4, N_5)$$
$$= [(1 - M_2N_1)(1 - M_2N_2)(1 - M_4N_3)(1 - M_4N_4)(1 - M_5N_5)]^{-1}.$$

It follows that the GF for SO(11) polynomial tensors based on a tensor (0, 0, 0, 0, 1) is

$$F(U; N_1, N_2, N_3, N_4, N_5) = [(1 - U^4)(1 - U^2N_1)(1 - U^2N_2)(1 - U^4N_3)(1 - U^4N_4)$$
$$\times (1 - UN_5)(1 - U^3N_5)]^{-1}.$$

$f_p = 7$, which agrees with $F(U; N)$. For the case $k = 1$ and $n = 6$ we get $f_p = 22$. When $n = 3$, $k = 2$ we have $f_p = 23$; things soon become difficult to evaluate.

3.7. SO(2n+1)×SO(2m+1) polynomial tensors based on a tensor (0, 0, ..., 0, 1)× (0, 0, ..., 0, 1)

The irreducible embedding is

$$B_n \cdot B_m \subset D_{n+m+1} \qquad n \geqslant 1, m \geqslant 1, n + m \geqslant 4$$

$$(0, 0, \ldots, 1) \times (0, 0, \ldots, 1) \subset (0, \ldots, 1, 0).$$

We consider the case $n=m=2$. The GF for SO(10) polynomial tensors based on a tensor (0, 0, 0, 1, 0) is

$$J(U;M_1,M_4)=[(1-U^2M_1)(1-UM_4)]^{-1}. \tag{3.13}$$

The GF for branching rules from SO(10) to SO(5) × SO(5) is

$$\begin{aligned}K(M_1,M_4;N_1,N_2,N_3,N_4)\\ &=[(1-M_1N_1)(1-M_1N_3)(1-M_1^2)(1-M_4N_2N_4)(1-M_4^2N_1N_3)\\ &\quad\times(1-M_4^2)(1-M_1M_4N_2N_4)]^{-1}\\ &\quad\times[(1-M_1M_4^2N_2^2N_3)^{-1}+M_1M_4^2N_1N_4^2(1-M_1M_4^2N_1N_4^2)^{-1}]\end{aligned}$$

where M_1 and M_4 carry the first and fourth labels of SO(10); N_1 and N_2 carry the first and second labels of the first SO(5); N_3 and N_4 carry the first and second label of the other SO(5). Following the prescription (1.2), the GF for SO(5) × SO(5) polynomial tensors based on a tensor (01) × (01) is

$$\begin{aligned}F(U;N_1,N_2,N_3,N_4)\\ &=[(1-U^2N_1)(1-U^2N_3)(1-U^4)(1-UN_2N_4)(1-U^2N_1N_3)(1-U^2)\\ &\quad\times(1-U^3N_2N_4)]^{-1}[(1-U^4N_2^2N_3)^{-1}+U^4N_1N_4^2(1-U^4N_1N_4^2)^{-1}]\end{aligned} \tag{3.14}$$

where U carries the degree as exponent.

Denoting an element of the integrity basis of (3.72) by $(u;\,n_1n_2;\,n_3n_4)$ where u is the degree (exponent of U) and n_1, n_2 and n_3, n_4 are the SO(5) labels; the full integrity basis is easily read from (3.14):

$$\begin{array}{lll} a=(2;1,0;0,0) & b=(2;0,0;1,0) & c=(2;0,0;0,0)\\ d=(4;0,0;0,0) & e=(1;0,1;0,1) & f=(2;1,0;1,0)\\ g=(3;0,1;0,1) & h=(4;0,2;1,0) & i=(4;1,0;0,2)\end{array}$$

with the product h^*i forbidden.

3.8. *Sp(8) polynomial tensors based on a (0, 1, 0, 0) tensor*

The irreducible embedding is

$$C_4\subset E_6$$

$$(0,1,0,0)\subset(0,0,0,0,1,0).$$

The GF for E_6 polynomial tensors based on a (0, 0, 0, 0, 1, 0) tensor is

$$J(U;M_1,M_5)=[(1-U^3)(1-UM_5)(1-U^2M_1)]^{-1}$$

where U carries the degree and M_1 and M_5 the E_6 labels. The GF for branching rules from E_6 to Sp(8) is

$$
\begin{aligned}
&K(M_1, M_5; N_1, N_2, N_3, N_4)\\
&\quad=[(1-M_1N_2)(1-M_5N_2)(1-M_1^2N_4)(1-M_5^2N_4)(1-M_1^2)\\
&\quad\times(1-M_5^2)(1-M_1^2M_5N_1N_3)(1-M_1M_5^2N_1N_3)]^{-1}\\
&\quad\times\{[(1-M_1M_5N_1N_3)(1-M_1^2M_5^2N_1^2N_4)]^{-1}\\
&\quad+M_1^2M_5^2N_3^2[(1-M_1M_5N_1N_3)(1-M_1^2M_5^2N_3^2)]^{-1}\\
&\quad+M_1M_5N_2[(1-M_1M_5N_2)(1-M_1^2M_5^2N_1^2N_4)]^{-1}\\
&\quad+M_1^3M_5^3N_2N_3^2[(1-M_1M_5N_2)(1-M_1^2M_5^2N_3^2)]^{-1}\}.
\end{aligned}
$$

Following the prescription (1.2) we get the GF for Sp(8) polynomial tensors based on (0, 1, 0, 0):

$$
\begin{aligned}
&F(U; N_1, N_2, N_3, N_4)\\
&\quad=[(1-U^3)(1-UN_2)(1-U^2N_2)(1-U^2N_4)(1-U^4N_4)(1-U^2)(1-U^4)\\
&\quad\times(1-U^4N_1N_3)(1-U^5N_1N_3)]^{-1}\{[(1-U^3N_1N_3)(1-U^6N_1^2N_4)]^{-1}\\
&\quad+U^6N_3^2[(1-U^3N_1N_3)(1-U^6N_3^2)]^{-1}+U^3N_2[(1-U^3N_2)\\
&\quad\times(1-U^6N_1^2N_4)]^{-1}+U^9N_2N_3^2[(1-U^3N_2)(1-U^6N_3^2)]^{-1}\}. \qquad (3.15)
\end{aligned}
$$

Note that $f_p=11$ ($r_h=36$, $l_h=4$, $r_j=l_j=0$), which agrees with (3.15). The integrity basis is read from (3.15); denoting an element of this basis by $(u; n_1, n_2, n_3, n_4)$, with u being the degree and the n_i being the Sp(8) labels, we have

$$
\begin{array}{ll}
a=(1;0,1,0,0) & a^*=(2;0,1,0,0)\\
b=(2;0,0,0,1) & b^*=(4;0,0,0,1)\\
c=(2;0,0,0,0) & c^*=(4;0,0,0,0)\\
d=(3;1,0,1,0) & e=(3;0,1,0,0)\\
f=(4;1,0,1,0) & f^*=(5;1,0,1,0)\\
g=(6;2,0,0,1) & h=(6;0,0,2,0)\\
i=(3;0,0,0,0) &
\end{array}
$$

with the products de and gh forbidden.

4. Polynomial tensors based on a tensor and a scalar

4.1. *Sp(6) polynomial tensors based on a tensor (0, 1, 0)*

Embeddings of the type $(n)+(\text{scalar})\subset(m)$ can also be exploited to construct a GF for polynomial tensors based on a tensor (n); the strategy consists in first constructing a GF $F(U; N)$ for tensors based on a tensor $(n)+(\text{scalar})$; the presence of the scalar manifests itself in $F(U; N)$ through a single denominator term $(1-U)$, which may be ignored if one is interested in polynomial tensors based on (n).

The embedding of interest is

$$\mathrm{Sp}(6) \subset \mathrm{SU}(6)$$

$$(0,1,0)+(0,0,0) \subset (0,1,0,0,0).$$

The GF $J(U; M)$ for SU(6) polynomial tensors based on a tensor $(0,1,0,0,0)$ is given by (2.3):

$$J(U; M_2, M_4, M_6) = [(1-U^3)(1-UM_2)(1-U^2M_4)]^{-1}.$$

The appropriate GF $K(M; N)$ for branching rules from SU(6) to Sp(6) is (Couture and Sharp 1980)

$$K(M_2, M_4; N_1, N_2, N_3) = [(1-M_2)(1-M_2N_2)(1-M_4)(1-M_4N_2)(1-M_2M_4N_1N_3)]^{-1}.$$

It follows that the GF for Sp(6) polynomial tensors based on a tensor $(010)+(000)$ is

$$F(U; N_1, N_2, N_3) = [(1-U)(1-U^3)(1-UN_2)(1-U^2)(1-U^2N_2)(1-U^3N_1N_3)]^{-1}. \tag{4.1}$$

The GF for Sp(6) polynomial tensors based on a tensor $(0,1,0)$ is obtained from (4.1) by removing the denominator factor $(1-U)$.

4.2. Sp(8) polynomial tensors based on a tensor (0, 1, 0, 0)

This GF has been evaluated in § 3.8 but here we show how to obtain it by considering the following embedding:

$$\mathrm{Sp}(8) \subset \mathrm{SU}(8)$$

$$(0,1,0,0)+(0,0,0,0) \subset (0,1,0,0,0,0,0).$$

The GF for SU(8) polynomial tensors based on a tensor $(0,1,0,0,0,0,0)$ is given in (2.3):

$$J(U; M_2, M_4, M_6) = [(1-UM_2)(1-U^2M_4)(1-U^3M_6)(1-U^4)]^{-1}.$$

The appropriate GF for branching rules from SU(8) to Sp(8) is

$$\begin{aligned} K(M_2, M_4, M_6; N_1, N_2, N_3, N_4) &= [(1-M_2N_2)(1-M_4N_4)(1-M_2M_6N_4)(1-M_4)(1-M_2M_4N_1N_3) \\ &\quad \times (1-M_6N_2)(1-M_4M_6N_1N_3)(1-M_6)(1-M_2)]^{-1} \\ &\quad \times \{(1-M_4N_2)(1-M_2M_4M_6N_1^2N_4) \\ &\quad + M_2M_4M_6N_3^2[(1-M_4N_2)(1-M_2M_4M_6N_3^2)]^{-1} \\ &\quad + M_2M_6N_1N_3[(1-M_2M_6N_1N_3)(1-M_2M_4M_6N_1^2N_4)]^{-1} \\ &\quad + M_2^2M_4M_6^2N_1N_3^3[(1-M_2M_6N_1N_3)(1-M_2M_4M_6N_3^2)]^{-1}\}. \end{aligned} \tag{4.2}$$

The integrity basis for these branching rules is found from (4.2); denoting an element of this basis by $(m_2, m_4, m_6; n_1, n_2, n_3, n_4)$ we get

$$\begin{array}{ll} a=(1,0,0;0,1,0,0) & a^*=(0,0,1;0,1,0,0) \\ b=(1,0,0;0,0,0,0) & b^*=(0,0,1;0,0,0,0) \\ c=(0,1,0;0,0,0,1) & d=(0,1,0;0,1,0,0) \\ e=(0,1,0;0,0,0,0) & g=(1,0,1;0,0,0,1) \\ f=(1,0,1;1,0,1,0) & h=(1,1,0;1,0,1,0) \\ h^*=(0,1,1;1,0,1,0) & i=(1,1,1;0,0,2,0) \\ j=(1,1,1;2,0,0,1) & \end{array}$$

with the products df and ij forbidden. If we follow the prescription described in (2.1) and discard the factor $(1-U)$, the GF (3.15) is obtained after some algebra.

5. SU(4) polynomial tensors based on a tensor (1, 0, 1); the group–subgroup orbit–orbit generating function

The solution here gives the reduction of the SU(4) enveloping algebra, discussed in § 3.5 and by Couture and Sharp (1980). Here we get the same result by using the embedding

$$\begin{array}{l} \mathrm{SU}(4)<\mathrm{Sp}(6) \\ (1,0,1)-(0,0,0)<(0,1,0) \end{array}$$

which is actually a subjoining (Patera *et al* 1980).

The GF for Sp(6) polynomial tensors based on a tensor (0, 1, 0) is

$$J(U; M_1, M_2, M_3) = [(1-U^2)(1-U^3)(1-UM_2)(1-U^2M_2)(1-U^3M_1M_3)]^{-1}. \tag{5.1}$$

The GF for branching rules from Sp(6) to SU(4) is

$$\begin{aligned} K(M_1, M_2, M_3; N_1, N_2, N_3) &= (1+M_1M_2N_2)[(1-M_1N_2)(1-M_1^2)(1+M_2)(1-M_2^2N_2^2)(1-M_3N_1^2) \\ &\quad \times(1-M_3N_3^2)]^{-1}[(1-M_2N_1N_3)^{-1}-M_3N_2(1+M_3N_2)^{-1}]. \end{aligned} \tag{5.2}$$

To find branching rules for a particular Sp(6) representation, the Sp(6) > SU(4) orbit-orbit GF was very useful. It is

$$\begin{aligned} O(M_1, M_2, M_3; N_1, N_2, N_3) &= [(1-M_1N_2)(1-M_2N_1N_3)]^{-1}[(1-M_3N_1^2)^{-1} \\ &\quad +M_3N_3^2(1-M_3N_3^2)^{-1}]. \end{aligned} \tag{5.3}$$

In the power expansion of (5.3) the occurrence of a term $M_1^{m_1}M_2^{m_2}M_3^{m_3}N_1^{n_1}N_2^{n_2}N_3^{n_3}$ implies the presence of the SU(4) (Weyl) orbit (n_1, n_2, n_3) in the Sp(6) orbit (m_1, m_2, m_3). A similar orbit-orbit GF can be determined for any group-subgroup (or group-subjoined-group). When the GF (5.2) is substituted into (5.1), one gets the GF for the SU(4) enveloping algebra, with an additional denominator factor $(1+U)$ due to the presence of the SU(4) scalar in the original embedding.

6. The reduction of the enveloping of SO(8)

The task of establishing the integrity basis and its syzygies can be quite formidable in cases where the basis is large; most 'tractable' cases have been worked out and what remains is much more difficult. If needed, these difficult cases may nevertheless be tackled by exploiting the irreducible embeddings discussed in this paper and making use of the computer; in such cases, one may at least obtain part of the integrity basis (which may be all one needs). As an example, we consider the problem of the reduction of the SO(8) enveloping algebra.

The structure of the SO(8) enveloping algebra is of interest in its own right. Recently, within the context of the development of the microscopic theory of nuclear collective motion, SO(8)* and SO(6, 2) were shown (Le Blanc and Rowe 1986) to be candidates for a dynamical group of SU(3); the tensor structure of SU(3) has been studied within these models and a complete set of SU(3) tensor operators (Biedenharn and Flath 1984, Le Blanc and Rowe 1987) has been defined in the enveloping algebras of SO(8)* and SO(6, 2); both these enveloping algebras are isomorphic to that of SO(8). Later in this section, we shall discuss some of their results.

Since the problem of the reduction of an enveloping algebra is equivalent (this follows from the Poincaré-Birkhoff-Witt theorem) to that of finding an integrity basis for polynomial tensors based on a tensor Γ that transform by the adjoint representation, we consider the following irreducible embedding (see § 3.4):

$$\begin{aligned}&\mathrm{SO}(8)\subset \mathrm{SU}(8)\\&(0,1,0,0)\subset(0,1,0,0,0,0,0).\end{aligned}\tag{6.1}$$

The generating function $J(U;\, M)$ for SU(8) polynomial tensors based on Γ is obtained from (2.3):

$$J(U;\, M_2, M_4, M_6)=[(1-U^4)(1-UM_2)(1-U^2M_4)(1-U^3M_6)]^{-1}\tag{6.2}$$

where M_2, M_4 and M_6 carry as exponents the second, fourth and sixth SU(8) representation labels and U carries the degree of the tensor.

The next step consists of establishing the integrity basis for the branching rules of SU(8) to SO(8). The integrity basis is large; up to (and including) degree 12, it consists of 81 elementary multiplets and 197 syzygies; we estimate that the full integrity basis contains more than 100 elementary tensors. As we go higher in degree the task of taking all possible products of powers of the known elementary multiplets soon becomes extremely tedious. The program MULTI does these products (automatically excluding the forbidden ones) and checks whether we have the correct total dimension and second-order index. This program is very flexible and can be used for any group-subgroup combination; it may also be used to calculate the integrity basis for the subjoining of a group to another one. The problem of guessing the new elementary multiplets and syzygies proved to be very difficult past degree 7. Fortunately branching rules can be obtained by Young diagram techniques (Cummins 1987); we wrote a program based on these techniques which gave us the branching rules of SU(8) to SO(8). Combining this program with MULTI has made the guessing part easy. The use of the irreducible embedding (6.1) proved very useful since it breaks the problem into several simpler ones; in the case of degree 12, the problem breaks down into 29 simpler ones as may be verified by expanding (6.2). Given an element $b=(m_2, m_4, m_6;\, n_1, n_2, n_3, n_4)$ of the integrity basis for the branching rules of SU(8) to SO(8), where the m and n are the representation labels of SU(8) and SO(8) respectively, it follows from (6.2) that the corresponding element e of the integrity basis for the SO(8) enveloping algebra is $e=(u=m_2+2m_4+3m_6;\, n_1, n_2, n_3, n_4)$ where u is the degree of the elementary tensor e. We give the integrity basis up to degree 10:

$$\begin{array}{lll}
1=(1;0,1,0,0) & 2=(2;2,0,0,0) & 3=(2;0,0,0,0)\\
4=(2;0,0,0,2) & 5=(2;0,0,2,0) & 6=(3;0,1,0,0)\\
7=(3;1,0,1,1) & 8=(3;0,1,0,0) & 9=(4;0,2,0,0)\\
10=(4;2,0,0,0) & 11=(4;0,0,0,0) & 12=(4;0,0,0,0)\\
13=(4;1,0,1,1) & 14=(4;1,0,1,1) & 15=(4;0,0,0,2)\\
16=(4;0,0,2,0) & 17=(5;1,1,1,1) & 18=(5;1,0,1,1)\\
19=(5;0,1,0,0) & 20=(5;1,0,1,1) & 21=(5;1,0,1,1)\\
22=(6;2,0,0,0) & 23=(6;0,0,0,0) & 24=(6;0,0,2,2)\\
25=(6;2,1,0,0) & 26=(6;0,1,0,2) & 27=(6;0,1,2,0)\\
28=(6;0,0,2,0) & 29=(6;0,0,0,2) & 30=(6;2,0,0,2)\\
31=(6;2,0,2,0) & 32=(6;1,0,1,1) & 33=(6;1,0,1,1)\\
34=(7;1,1,1,1) & 35=(7;1,1,1,1) & 36=(7;1,0,1,1)\\
37=(7;1,0,1,1) & 38=(7;2,0,2,0) & 39=(7;2,0,0,2)\\
40=(7;0,0,2,2) & 41=(7;1,0,1,1) & 42=(8;0,1,2,2)\\
43=(8;1,0,1,1) & 44=(8;0,0,2,2) & 45=(8;2,1,0,0)\\
46=(8;0,2,0,0) & 47=(8;0,1,2,0) & 48=(8;0,1,0,2)\\
49=(8;2,1,0,2) & 50=(8;2,1,2,0) & 51=(8;2,0,0,2)\\
52=(8;2,0,2,0) & 53=(8;1,0,1,1) & 54=(9;1,1,1,1)\\
55=(9;1,1,1,1) & 56=(9;2,0,2,0) & 57=(9;2,0,0,2)\\
58=(9;0,0,2,2) & 59=(9;1,0,1,1) & 60=(9;2,0,2,0)\\
61=(9;2,0,0,2) & 62=(9;0,0,2,2) & 63=(10;0,0,2,2)\\
64=(10;2,1,0,0) & 65=(10;0,1,2,0) & 66=(10;0,1,0,2)\\
67=(10;2,0,2,0) & 68=(10;2,0,0,2) & 69=(10;2,1,2,0)\\
70=(10;2,1,0,2). & &
\end{array} \qquad (6.3)$$

The following products are forbidden: 13^2, 17^2, 18^2, 21^2, 8*17, 6*17, 4*31, 13*14, 7*25, 13*17, 13*20, 14*17, 15*17, 16*17, 14*18, 13*21, 13*25, 2*42, 17*18, 14*26, 14*27, 14*32, 14*33, 4*52, 5*51, 4*50, 5*49, 8*35, 17*19, 13*32, 13*33, 13*26, 14*25, 13*27, 17*21, 16*25, 15*25, 13*31, 13*30, 14*24 and 7*8*14.

All tensors obtained from the stretched product (example: $1^*4=(3; 0, 1, 0, 2)$) of these elementary tensors constitute elements of a basis for tensor operators in the enveloping algebra of SO(8). Kostant has shown that there always exists a representation in which any of these tensors exist (i.e. have non-vanishing matrix elements) and are functionally independent (if we exclude products with Casimirs).

However, if we consider the action of these operators on basis states of a representation for which one or more labels vanish, then this basis (as well as the integrity basis) may reduce considerably; some of these operators simply vanish or are no longer functionally independent. We say that the enveloping algebra is degenerate. The generating function $DE(U; N)$ describing a basis for a degenerate enveloping algebra of a group G may be obtained (Giroux *et al* 1984) from the generating function $B(N; M)$ for branching rules of $G \supset H \times U(1)^n$, where n is the number of vanishing representation labels of the states on which these operators are acting and H is the subgroup whose Dynkin diagram is obtained from that of G by removing vertices (and lines attached to them) corresponding to the non-vanishing labels. By keeping only the $H \times U(1)^n$ scalar part of $B(N; M)$ we get a generating function which corresponds to $DE(U; N)$ but with the variable U, which carries the degree, missing. As an example, let us consider the case where only the first label is non-zero. The Dynkin diagram corresponding to the zero labels is that of SO(6); the generating function $B(N, M)$ for branching rules of SO(8) to SO(6) × U(1) is

$$\begin{aligned}
&B(N_1, N_2, N_3, N_4; M_1, M_2, M_3, M_4) \\
&\quad=[(1-N_4M_1M_4)(1-N_4M_3M_4^{-1})(1-N_3M_1M_4^{-1})(1-N_3M_3M_4)(1-N_2) \\
&\qquad\times(1-N_2M_2M_4^2)(1-N_1M_2)(1-N_1M_4^{-2})]^{-1} \\
&\qquad\times\{[(1-N_2M_1M_3)(1-N_2M_2M_4^{-2})]^{-1}+N_1M_4^2[(1-N_2M_1M_3) \\
&\qquad\times(1-N_1M_4^2)]^{-1}+N_3N_4M_2[(1-N_3N_4M_2)(1-N_2M_2M_4^{-2})]^{-1} \\
&\qquad+N_1N_3N_4M_2M_4^2[(1-N_3N_4M_2)(1-N_1M_4^2)]^{-1}\}
\end{aligned}$$

where the N and M carry the SO(8) and SO(6) × U(1) representation labels respectively (M_4 carry the U(1) label); setting M_1, M_2 and M_3 equal to zero and keeping the zero degree part of M_4, we get

$$F(N)=[(1-N_2)(1-N_1^2)]^{-1}.$$

Knowledge of the integrity basis (6.3) of the non-degenerate enveloping algebra is useful in determining the degree of the tensors in the degenerate case. The highest degree (modulo products with Casimirs) u_h at which a tensor (n_1, n_2, n_3, n_4) appear in the non-degenerate enveloping algebra of SO(8) is equal to the sum of the coefficient of the simple roots in the highest weight of (n_1, n_2, n_3, n_4); we get $u_h = 3n_1+5n_2+3n_3+3n_4$. Obviously, the degree of the tensor (0, 1, 0, 0) has to be one. For (2, 0, 0, 0), $u_h=6$ and therefore the integrity basis (6.3) informs us that there are only three such tensors; based on past experience we expect that it is the lowest degree one that remains (of course, this remains to be proven). Thus, the generating function for the degenerate enveloping algebra is

$$DE(U; N_1, N_2)=[(1-UN_2)(1-U^2N_1^2)]^{-1}. \tag{6.4}$$

The integrity basis has therefore been reduced to two elementary tensors: (1; 0, 1, 0, 0) and (2; 2, 0, 0, 0).

The number of linearly independent irreducible tensor operators that transform under the action of SU(3) according to some given irreducible representation is equal to the dimension of this representation (Baird and Biedenharn 1965). Biedenharn and Flath (1984), and more recently Le Blanc and Rowe (1986, 1987), have given a realisation of such a complete set C in the enveloping algebra of SO(8). We now produce this result using generating function techniques.

The set C is defined within a subset of the full SO(8) enveloping algebra, namely the set of tensors defined by the following generating function:

$$F(U; N_2) = (1 - UN_2)^{-1}. \tag{6.5}$$

Restricting ourselves to SO(8) representations for which only the second label is non-zero, the GF for branching rules from SO(8) to SU(4) is

$$F1(N_2; M_1, M_2, M_3) = [(1 - N_2 M_1 M_3)(1 - N_2 M_2)^2 (1 - N_2)]^{-1} \tag{6.6}$$

where the M carry the SU(4) labels. The GF for branching rules from SU(4) to SU(3) is

$$F2(M_1, M_2, M_3; P_1, P_2)$$
$$= [(1 - M_1)(1 - M_1 P_1)(1 - M_2 P_1)(1 - M_2 P_2)(1 - M_3 P_2)(1 - M_3)]^{-1} \tag{6.7}$$

where the P carry the SU(3) labels. By combining (6.5)-(6.7) using the procedure described at (1.2) we get the following GF for SU(3) tensors in the set C:

$$F4(U; P_1, P_2)$$
$$= [(1 - U)^2 (1 - UP_1 P_2)(1 - UP_1)(1 - UP_2)]^{-1} [(1 - UP_1)^{-1}$$
$$+ UP_2 (1 - UP_2)^{-1}]^2. \tag{6.8}$$

The integrity basis corresponding to the GF given in (6.8) is

$1 = (1; 0, 0)$	$2 = (1; 0, 0)$	$3 = (1; 1, 1)$
$4 = (1; 1, 0)$	$5 = (1; 0, 1)$	$6 = (1; 1, 0)$
$7 = (1; 1, 0)$	$8 = (1; 0, 1)$	$9 = (1; 0, 1)$

with the products 6*8 and 7*9 being forbidden; the first label denotes the degree, and the second and third labels the SU(3) labels. A GF $MUL(P_1, P_2)$ giving the multiplicity of tensors in C (we discard the factors $(1 - U)$ in (6.8)) is obtained from (6.8) by setting $U = 1$; after some algebra we get

$$MUL(P_1, P_2) = (1 - P_1 P_2)[(1 - P_1)^3 (1 - P_2)^3]^{-1}. \tag{6.9}$$

The coefficient m_{ab} of a term $m_{ab} P_1^a P_2^b$ in the expansion of (6.9) gives the multiplicity of the tensor (a, b) in C. From the SU(3) character generator (Stanley 1980) one obtains (6.9) as the dimension GF for SU(3); the coefficient m_{ab} of a term $m_{ab} P_1^a P_2^b$ in the expansion of (6.9) is therefore equal to the dimension of the SU(3) representation (a, b). It follows that (6.8) defines a complete set of SU(3) tensor operators.

We conclude this section with a few comments on how one could use the subjoining $D_4 < C_4$ is obtain the GF for the SO(8) enveloping algebra. The embedding of interest is

$$(0, 1, 0, 0) - (0, 0, 0, 0) < (0, 1, 0, 0). \tag{6.10}$$

The GF for $D_4 < C_4$ is

$$\begin{aligned}
&K(M_1, M_2, M_3, M_4; N_1, N_2, N_3, N_4) \\
&\quad = [(1-M_1^2)(1-M_2^2N_1^2)(1-M_3^2N_2^2)(1-M_4N_3^2) \\
&\quad\quad \times(1-M_4N_4^2)]^{-1}\{[(1-M_1N_1)(1-M_2N_2)(1-M_3N_3N_4)]^{-1} \\
&\quad\quad -M_4N_2[(1-M_1N_1)(1-M_2N_2)(1+M_4N_2)]^{-1} \\
&\quad\quad -M_3N_1[(1-M_1N_1)(1+M_3N_1)(1-M_3N_3N_4)]^{-1} \\
&\quad\quad +M_3M_4N_1N_2[(1-M_1N_1)(1+M_3N_1)(1+M_4N_2)]^{-1} \\
&\quad\quad -M_2[(1+M_2)(1-M_2N_2)(1-M_3N_3N_4)]^{-1} \\
&\quad\quad +M_2M_4N_2[(1+M_2)(1-M_2N_2)(1+M_4N_2)]^{-1} \\
&\quad\quad +M_2M_3N_1[(1+M_2)(1+M_3N_1)(1-M_3N_3N_4)]^{-1} \\
&\quad\quad -M_2M_3M_4N_1N_2[(1+M_2)(1+M_3N_1)(1+M_4N_2)]^{-1}\} \qquad (6.11)
\end{aligned}$$

where the M_i carry the Sp(8) representation labels and the N_i those of SO(8). Denoting an element of the integrity basis by $(m_1, m_2, m_3, m_4; n_1, n_2, n_3, n_4)$ where the m and n are the representation labels of Sp(8) and SO(8) respectively, the integrity basis of (6.11) is

$$\begin{array}{ll}
1 = (1,0,0,0;1,0,0,0) & 2 = (2,0,0,0;0,0,0,0) \\
3 = (0,1,0,0;0,1,0,0) & 4 = -(0,1,0,0;0,0,0,0) \\
5 = (0,2,0,0;2,0,0,0) & 6 = (0,0,1,0;0,0,1,1) \\
7 = -(0,0,1,0;1,0,0,0) & 8 = (0,0,2,0;0,2,0,0) \\
9 = (0,0,0,1;0,0,2,0) & 10 = (0,0,0,1;0,0,0,2) \\
11 = -(0,0,0,1;0,1,0,0). &
\end{array}$$

The following products are forbidden: 1*4, 3*7 and 6*11. The desired GF is then obtained by 'substituting' (6.11) into (3.81); the presence of a scalar in (6.10) will manifest itself in the GF by a denominator term $(1+U)$ which may be discarded. If, in principle, the GF describing the decomposition of the SO(8) enveloping algebra may be obtained in this way, it is expected that the algebra would be quite tedious. We note that the tables of Bremmer *et al* (1985), giving the orbits contained in an irreducible representation, were very useful in evaluating (6.11).

7. Conclusion

The embeddings considered in § 3 constitute only part of Dynkin's list. Cases not mentioned in this paper can also be used to find the integrity basis for polynomial tensors; however, by calculating the number of denominator factors of such GF, it is clear that even the simplest cases are difficult and therefore the use of the computer is highly recommended. A program like MULTI (which one of us (MC) intends to submit for publication to the *Journal of Computer Physics Communications*), combined with a program such as SCHUR which gives the branching (among other things) for many group-subgroup pairs (now commercially available from SCHUR Software Associates, New Zealand), would allow one to tackle many of these more difficult cases.

Note that we have considered only polynomial tensors (symmetric plethysms). The simplification exploited here is equally effective in determining generating functions for general plethysms; in the chain $H \subset G \subset SU(d)$ of § 1, one would need branching rules for all representations of $SU(d)$ to G, and hence to H (Patera and Sharp 1980).

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